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Hamiltonicity and Minimum Degree in 3-connected Claw-Free Graphs

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Using Ryjáček's closure, we prove that any 3-connected claw-free graph of order v and minimum degree $\delta \geqslant \frac{v+38}{10}$ is hamiltonian. This improves a theorem of Kuipers and Veldman who got the same result with the stronger hypotheses $\delta \geqslant \frac{v+29}{8}$ and v sufficiently large and nearly proves their conjecture saying that the result holds as soon as $\delta \geqslant \frac{v+6}{10}$ for v sufficiently large. © 2001 Academic Press

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1. INTRODUCTION

The graphs $G = (V(G), E(G))$ we consider here are simple, unless otherwise stated, and finite. When G is a multigraph, it can have multiple edges but no loops. The degree and neighborhood of a vertex x of G are respectively denoted by $d_G(x)$ and $N_G(x)$. The minimum degree of G is $\delta(G)$. If $S \subseteq V(G)$, $G[S]$ is the subgraph induced in G by S. For a vertex x of G, $N_S(x) = N_G(x) \cap S$ and $d_S(x) = |N_S(x)|$. Two edges are adjacent if they have exactly one common vertex.

A *circuit* C of G, sometimes called a *closed trail*, is an eulerian subgraph, that is a connected even subgraph, of G. Given two circuits C and C' of G, the sum $C + C'$ is the subgraph of vertex set $V(C + C') = V(C) \cup V(C')$ and whose edge set $E(C + C')$ is the symmetric difference $(E(C) \setminus E(C')) \cup (E(C') \setminus E(C))$. The subgraph $C + C'$ is clearly even and is thus a circuit of G if and only if it is connected. The circuit C of G is *maximal* if no circuit of G has a vertex set strictly containing $V(C)$. The circuit is *dominating* if every edge of G has at least one extremity in $V(C)$ or, equivalently, if every

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component of $G - V(C)$ is an isolated vertex. A *cycle* of G is a circuit visiting exactly once each of its vertices.

Let H be the line graph $L(G)$ of a graph G. The order $\nu(H)$ of H is equal to the number $m(G)$ of edges of G, and $\delta(H) = \min\{d_G(x) + d_G(y) - 2; xy \in E(G)\}$. A cycle of H is the image of a circuit or of a star of G. We will use the well known following result.

THEOREM A (Harary and Nash-Williams [2]). *The line graph $H = L(G)$ of a graph G is hamiltonian if and only if G has a dominating circuit or is isomorphic to $K_{1,s}$ for some $s \geqslant 3$.*

Similarly, it is known that if $L(G)$ is k-connected, then G is essentially k-edge-connected, which means that the only edge-cutsets of G having less than k edges are the sets of edges incident to some vertex of G.

A graph H is a claw-free if it does not contain the star $K_{1,3}$ as an induced subgraph. Every line graph is claw-free. Moreover, the introduction of Ryjáček's closure in claw-free graphs showed that the study of many hamiltonian problems in claw-free graphs could be reduced to the same study for line graphs. The *closure* $\mathrm{cl}(H)$ of a claw-free graph H is obtained by recursively completing the neighborhood of any locally connected vertex of H, as long as this is possible. The closure $\mathrm{cl}(H)$ is a well-defined claw-free graph and its connectivity is at least equal to the connectivity of H. The following basic properties of the closure $\mathrm{cl}(H)$ were proved in [7].

THEOREM B (Ryjáček [7]). *Let H be a claw-free graph and* $\mathrm{cl}(H)$ *its closure. Then*

(i) *there is a triangle-free graph G such that* $\mathrm{cl}(H)$ *is the line graph of G,*

(ii) *both graphs H and* $\mathrm{cl}(H)$ *have the same circumference.*

Consequently, H is hamiltonian if and only if $\mathrm{cl}(H)$ is hamiltonian.

A claw-free graph H is said to be closed if it is equal to its closure, in other terms if it is the line graph $L(G)$ of a triangle-free graph G.

Many works have been done to give sufficient conditions for a claw-free graph H to be hamiltonian in terms of its minimum degree $\delta(H)$. These conditions depend on the connectivity $\kappa(H)$. If $\kappa(H) = 4$, Matthews and Sumner [6] conjectured that H is hamiltonian and this conjecture has been neither proved nor disproved so far. For connectivity 2 or 3, the most recent results are the following.

When $\kappa(H) = 2$, Kuipers and Veldman [5], and independently Favaron, Flandrin, Li and Ryjáček [1], proved that every 2-connected claw-free graph H of sufficiently large order $\nu(H)$ and such that $\delta(H) > \frac{\nu(H) + c}{6}$ (where

c is a constant) is hamiltonian except if H is a member of ten well-defined families of graphs.

When $\kappa(H)=3$, Kuipers and Veldman established the following result.

THEOREM C (Kuipers and Veldman [5]). *Let H be a 3-connected claw-free graph of order v with $\delta(H) \geqslant \frac{v+29}{8}$. If v is sufficiently large, then H is hamiltonian.*

They also proposed a stronger conjecture.

Conjecture D (Kuipers and Veldman [5]). Let H be a 3-connected claw-free graph of order v with $\delta(H) \geqslant \frac{v+6}{10}$. If v is sufficiently large, then H is hamiltonian.

In this paper we nearly prove this conjecture by showing

THEOREM 1. *If H is a 3-connected claw-free graph of order v and minimum degree δ such that $\delta > \frac{v+37}{10}$, then H is hamiltonian.*

2. PRELIMINARY LEMMAS

LEMMA 1. *Let S be a set of vertices of a graph G contained in a circuit of G and let C be a maximal circuit of G containing S. Assume that some component A of $G-V(C)$ is not an isolated vertex and is related to C by at least r edges. Then*

1. *G contains a matching T of $r+1$ edges such that at most $2r$ edges of G are adjacent to two distinct edges of T.*

2. *The number $m(G)$ of edges of G is related to the minimum degree $\delta(H)$ of the line graph H of G by $m(G) \geqslant (r+1)\,\delta(H)-r+1$.*

Proof 1. If some vertex x of C has two different neighbors y and z in A, then if C' is the cycle $xyP_{yz}zx$ where P_{yz} is a path of $G[A]$ joining y and z, the circuit $C+C'$ contradicts the maximality of C. Hence there are at least r vertices of C adjacent to some vertex of A. We choose an arbitrary orientation of the circuit C. It induces a set of transitions at each vertex of C and an orientation of each edge. If a_1 is the endvertex on C of an edge between A and C, we choose a successor b_1 of a_1 on the oriented circuit C and describe C following its orientation. Let $a_2, ..., a_r$ be the extremities on C, encountered in this order, of $r-1$ other edges between A and C. For $2 \leqslant i \leqslant r-1$, let b_i be the successor of a_i. Since $G[A]$ is connected, there exists, for each pair of different indices i and j, a path P_{ij} between a_i and a_j whose internal vertices are in A. Let moreover $e_0 = yz$ be an edge of $G[A]$.

CLAIM 1. *The $r+1$ edges a_ib_i, $1 \leqslant i \leqslant r$, and e_0 form a matching T of G.*

Proof of Claim 1. By its definition, e_0 is not adjacent to any a_ib_i. All the vertices a_i are distinct. Suppose $a_i = b_j$ for some $i \neq j$. Then, if we put $C' = a_iP_{ij}a_ja_i$, the circuit $C + C'$, obtained from C by replacing the edge a_jb_j by the path P_{ij}, contradicts the maximality of C. Suppose $b_i = b_j$ for some $i \neq j$ and let $C' = a_iP_{ij}a_jb_ia_i$. By the orientation of a_ib_i and a_jb_i, the circuit C contains a path between b_i and a_i, and a path between b_j and a_j, avoiding the edges a_ib_i and a_jb_i. Hence the deletion of these two edges does not disconnect C at b_i, and thus $C + C'$ is a circuit contradicting the maximality of C. ∎

CLAIM 2. *There are at most r edges joining two different edges a_ib_i, $1 \leqslant i \leqslant r$.*

Proof of Claim 2. Let f be an edge of G joining a_ib_i to a_jb_j.

Case 1: f is not an edge of C. Consider the cycle $C' = a_iP_{ij}a_ja_i$ if $f = a_ia_j$, $C' = a_iP_{ij}a_jb_ja_i$ if a_ib_j (and symmetrically $C' = a_iP_{ij}a_jb_ia_i$ if $f = a_jb_i$), and $C' = a_iP_{ij}a_jb_jb_ia_i$ if $f = b_ib_j$. When $f \neq b_ib_j$, $C + C'$ contains all the edges of C except at most one and thus it is connected. In the last case, $C + C'$ is also connected since, from the orientation of a_ib_i and a_jb_j, C contains a path from b_i to a_j, and a path from b_j to a_i, avoiding both edges a_ib_i and a_jb_j. In any case $C + C'$ contradicts the maximality of C.

Case 2: f is an edge of C. The edge f cannot be of type a_ia_j for otherwise, if $C' = a_iP_{ij}a_jfa_i$, the circuits C and C' have one common edge and thus $C + C'$ contradicts the maximality of C. Hence f is of the kind b_ia_j or b_ib_j. For each edge a_ib_i, there exists at most one other edge $f(a_ib_i)$ of C having one endvertex in b_i and such that $\{a_ib_i, f(a_ib_i)\}$ is an edge-cutset of C (such an edge can exist only if b_i either has degree 2 in C or is a cut-vertex of C). We denote by F the set of the edges $f(a_ib_i)$ which exist, $1 \leqslant i \leqslant r$. The set F has at most r elements.

If $f = b_ia_j$, let $C' = a_iP_{ij}a_jb_ia_i$. In the circuit $C + C'$, the two edges a_ib_i and $f = b_ia_j$ of C are replaced by the path P_{ij}. Hence $C + C'$ is connected unless $\{a_ib_i, f\}$ is an edge-cutset of C, that is, unless $f = f(a_ib_i)$.

If $f = b_ib_j$, let $C' = a_iP_{ij}a_jb_jb_ia_i$. In the circuit $C + C'$, the three edges a_ib_i, a_jb_j and $f = b_ib_j$ of C are replaced by the path P_{ij}. Suppose f oriented from b_i to b_j. The circuit C contains a path from b_j to a_i avoiding the three edges a_ib_i, a_jb_j and f. Hence $C + C'$ is connected unless $\{a_ib_i, f\}$ is an edge-cutset of C, that is, unless $f = f(a_ib_i)$.

By the maximality of C, $C + C'$ is not connected and thus f belongs to F. Therefore there are at most $|F| \leqslant r$ edges joining two different edges a_ib_i. ∎

CLAIM 3. *There are at most r edges between e_0 and the edges $a_i b_i$, $1 \leqslant i \leqslant r$.*

Proof of Claim 3. We have already seen that no vertex of C can be adjacent to both extremities y and z of e_0. Suppose for instance a_i adjacent to y and b_i adjacent to z. Then, if $C' = a_i y z b_i a_i$, the circuit $C + C'$ contradicts the maximality of C. Hence for each i, there is at most one edge joining e_0 to the edge $a_i b_i$. ∎

Claims 1, 2 and 3 prove the first part of Lemma 1.

2. There are at least $|T|\,\delta(H)$ edges of G not belonging to T and adjacent to at least one edge of T. Among them, at most $2r$ are adjacent to two different edges of T and are thus counted twice. Hence $m(G) \geqslant |T|\,\delta(H) - 2r + |T|$ with $|T| = r+1$. Therefore $m(G) \geqslant (r+1)\,\delta(H) - r + 1$. ∎

To prove the next lemma, we need the following result, independently proved by Holton, MacKay, Plummer and Thomassen, and by Kelmans and Lomonosov.

THEOREM D ([3], [4]). *In a 3-regular 3-connected graph any 9 vertices lie on a common cycle.*

LEMMA 2. *In an 3-edge-connected multigraph any 9 vertices lie on a common circuit.*

Proof. Let $\mathscr{G}'$ be a simple graph obtained from the 3-edge-connected multigraph $\mathscr{G}$ by replacing each vertex x of degree $d(x)$ by a cycle $C(x)$ of order $d(x)$. If two vertices x and y are adjacent in $\mathscr{G}$, we replace the edge xy of $\mathscr{G}$ by an edge, called a blue edge, between a vertex of $C(x)$ and a vertex of $C(y)$ in such a way that the blue edges form a perfect matching of $\mathscr{G}'$. The new graph $\mathscr{G}'$ is cubic and 3-edge-connected unless $\mathscr{G}$ admits a cutvertex x with $d(x) \geqslant 6$ since then some two edges of $C(x)$ could form a 2-edge-cutset of $\mathscr{G}'$. In this case we define the edges of $C(x)$ so as to avoid this situation. Let $\mathscr{C}_i$, $1 \leqslant i \leqslant p$, be the components of $G - \{x\}$ and let $v_{i,1}, v_{i,2}, ..., v_{i,q_i}$ be the neighbors of x in $\mathscr{C}_i$. Since G is 3-edge-connected, $q_i \geqslant 3$ for all i. Let $x_{i,j}$ be the vertex of $C(x)$ adjacent by a blue edge of $\mathscr{G}'$ to a vertex $w_{i,j}$ of $C(v_{i,j})$. We define $C(x)$ as $x_{1,1} x_{2,1} x_{3,1} \cdots x_{p,1} x_{1,2} x_{2,2} x_{3,2} \cdots x_{p,2} x_{1,3} x_{1,4} \cdots x_{1,q_1} x_{2,3} x_{2,4} \cdots x_{2,q_2} x_{3,3} x_{3,4} \cdots x_{3,q_3} \cdots x_{p,3} x_{p,4} \cdots x_{p,q_p} x_{1,1}$. Then no pair of edges of $C(x)$ forms an edge-cutset of $\mathscr{G}'$. Hence, if we choose $C(x)$ in this way at each cutvertex x of $\mathscr{G}$, the graph $\mathscr{G}'$ is 3-edge-connected. As it is cubic, it is also 3-connected.

Let now $S = \{y_1, y_2, ..., y_q\}$ be a set of at most 9 vertices of $\mathscr{G}$. For each vertex y_i we choose a vertex y'_i of $\mathscr{G}'$ in $C(y_i)$. The set $S' = \{y'_1, y'_2, ..., y'_q\}$

is contained in a cycle C of $\mathcal{G}'$ by Theorem D. If we contract each cycle $C(x)$ of $\mathcal{G}'$ to recover the multigraph $\mathcal{G}$, the cycle C of $\mathcal{G}'$ yields a circuit of $\mathcal{G}$ containing S. ▮

3. PROOF OF THEOREM 1

By Theorem B, the graph H is hamiltonian if and only if its closure $\mathrm{cl}(H)$ is hamiltonian. As $\nu(\mathrm{cl}(H)) = \nu(H)$, $\delta(\mathrm{cl}(H)) \geqslant \delta(H)$, and $\mathrm{cl}(H)$ is 3-connected, the graph $\mathrm{cl}(H)$ satisfies the same hypotheses as H. Hence it is sufficient to prove Theorem 1 for closed claw-free graphs.

Consider therefore the line graph H of a triangle-free graph G, and suppose that H is 3-connected and satisfies $\delta(H) > \frac{\nu(H)+37}{10}$. Assume by contradiction that H is not hamiltonian. By Theorem A, the graph G contains no dominating circuit.

Let B be the subset of the vertices of G of degree 1 or 2. Since H is 3-connected, every edge of G has degree at least 3 and thus the set B is independent. Let $X_0 = N_G(B)$. We name the vertices of X_0 as $x_1, x_2, ..., x_p$ in the following way. Assume the vertices $x_1, ..., x_i$ are already defined or else put $i = 0$. Let y_{i+1} denote a vertex of B which is adjacent to some vertex of $X_0 \backslash \{x_1, ..., x_i\}$. Either y_{i+1} has exactly one neighbor in $X_0 \backslash \{x_1, ..., x_i\}$ and we name it x_{i+1}. Or y_{i+1} has exactly two neighbors in $X_0 \backslash \{x_1, ..., x_i\}$ and we name them x_{i+1} and x_{i+2} and put $y_{i+2} = y_{i+1}$. Let $Y_0 = \{y_1, ..., y_p\}$. We note that if $1 \leqslant i < j \leqslant p$, then $y_i y_j$ and $y_i x_j$ are not edges of G, except for the edges $y_i x_{i+1}$ when $y_i = y_{i+1}$; and that the components of the subgraph induced by the edges $x_i y_i$, $1 \leqslant i \leqslant p$, are paths of length 1 or 2.

Consider now a matching M of G formed by $q-p$ edges $x_i y_i$ of G, $p+1 \leqslant i \leqslant q$, considered in this order and such that

(i) the sets X_0, Y_0, $X = \{x_{p+1}, ..., x_q\}$ and $Y = \{y_{p+1}, ..., y_q\}$ are pairwise disjoint

(ii) for $p+1 \leqslant i < j \leqslant q$, $y_i y_j$ and $y_i x_j$ are not edges of G.

We choose this matching as large as possible subject to the conditions (i) and (ii). Note that by the definition of X_0 and Y_0, the whole set B is disjoint from $X \cup Y$ and that Property (ii) holds for any $1 \leqslant i < j \leqslant q$.

Let J be the set of indices j between $p+1$ and q such that y_j is adjacent to some vertex $z \notin X_0 \cup Y_0 \cup X \cup Y$ with $y_k z \notin E(G)$ for $1 \leqslant k < j$. For each $j \in J$ we choose such a vertex z_j and we put $I = \{p+1, p+2, ..., q\} \backslash J$. Let $X_I = \{x_i \in X; i \in I\}$, $X_J = \{x_i \in X; i \in J\}$, $Y_I = \{y_i \in Y; i \in I\}$ and $Y_J = \{y_i \in Y; i \in J\}$.

CLAIM. *The set $S = X_0 \cup X_I \cup Y_J$ is not contained in any circuit of G.*

Proof of the claim. Suppose the claim false and let C be a maximal circuit of G containing $S = X_0 \cup X_I \cup Y_J$ and $R = V(G) \backslash V(C)$. By the assumption that G has no dominating circuit, at least one component A of $G[R]$ is not a single vertex. This component A is disjoint from Y_0 since the vertices of Y_0 are isolated in $G[R]$.

Suppose first that every vertex of A has a neighbor in C. Then, if uv is an edge of A and if s denotes the number of edges between A and C, $s \geqslant d_C(u) + d_C(v) + |A| - 2$. Since G is triangle-free, $d_A(u) + d_A(v) \leqslant |A|$ and thus $d_G(u) + d_G(v) = d_C(u) + d_C(v) + d_A(u) + d_A(v) \leqslant d_C(u) + d_C(v) + |A|$. Hence $s \geqslant d_G(u) + d_G(v) - 2 \geqslant \delta(H)$. By Lemma 1 applied with $r = \delta(H)$, the number of edges of G satisfies $m(G) \geqslant \delta(H)^2 + 1$. This contradicts $m(G) = v(H) < 10\delta(H) - 37$.

Therefore A contains a vertex z such that $N_G(z) \subseteq A$. Then $z \notin X_0 \cup Y_0 \cup X \cup Y$ and the neighbors of z are all in $Y_I \cup X_J \cup (R \backslash Y_0 \cup Y_I \cup X_J)$.

If z has a neighbor in Y_I, let i be the least index such that $y_i \in Y_i$ and $zy_i \in E(G)$. Since z has no neighbor in Y_J, $zy_k \notin E(G)$ for all $k < i$, in contradiction to the definition of I. Hence z has no neighbor in Y_I, and thus in Y.

If z has a neighbor in X_J, let x_j be the vertex of $N_G(z) \cap X_J$ with the largest index. Consider the ordered sets $X' = \{x_{p+1}, ..., x_{j-1}, x_j, z_j, x_{j+1}, ..., x_q\}$ and $Y' = \{y_{p+1}, ..., y_{j-1}, z, y_j, y_{j+1}, ..., y_q\}$. The vertex z is neither adjacent to any x_k with $k > j$ by the definition of x_j and since z has no neighbor in X_I, nor to any vertex of Y as said above. The vertex z_j is not adjacent to any vertex y_k with $k < j$ by the choice of z_j. If $zz_j \notin E(G)$, then the sets X' and Y' define a matching M' which satisfies (i) and (ii), and thus which contradicts the maximality of M. If $zz_j \in E(G)$, then the circuit $C + C'$, with $C' = y_j z_j z x_j y_j$, satisfies $V(C') \cap V(C) = \{y_j\}$ since z has no neighbor in C, and thus contradicts the maximality of C. Hence $N_G(z) \cap X_J = \varnothing$ and z has no neighbor in X.

Finally if z has a neighbor t in $R \backslash (Y_0 \cup Y_I \cup X_J)$, then the matching M'' corresponding to the ordered sets $X'' = \{t, x_{p+1}, ..., x_q\}$ and $Y'' = \{z, y_{p+1}, ..., y_p\}$ satisfies the conditions (i) and (ii) since z has no neighbor in $X \cup Y$. This contradicts the maximality of M and achieves the proof of the claim. ▮

Let $\mathscr{G}$ be the graph or multigraph obtained from G by deleting the vertices of degree 1 or 2 and replacing each path $\xi y \eta$ where $d_G(y) = 2$ by the edge $\xi\eta$. Since G is essentially 3-edge-connected, $\mathscr{G}$ is 3-edge-connected. Moreover, to each circuit C of $\mathscr{G}$ corresponds a circuit of G containing $V(C)$. Since $S \cap B = \varnothing$, the set S is contained in $V(\mathscr{G})$. And since it is contained in no circuit of G by the previous claim, S is contained in no circuit of $\mathcal{G}$. By Lemma 2, the set S has more than 9 vertices and thus $q \geqslant 10$. Let $F = \{x_i y_i; 1 \leqslant i \leqslant 10\}$, $P = \{x_i; 1 \leqslant i \leqslant 10\}$ and $Q = \{y_i; 1 \leqslant i \leqslant 10\}$. We suppose that F consists of l paths of length 2 with $0 \leqslant l \leqslant 5$ and $10 - 2l$ edges of a matching. Then $|P| = 10$ and $|Q| = 10 - l$. We know that Q is independent, that $y_i x_j \notin E(G) \backslash F$ for any $y_i \in Q$ and $x_j \in P$ with $1 \leqslant i < j \leqslant 10$, and that G is triangle-free. Hence, two different edges of F are joined by at most one edge of G which is of type $x_i x_j$ or $x_i y_j$ with $1 \leqslant i < j \leqslant 10$. More precisely, we can give an upper bound on the number μ of edges of G which are adjacent to two different edges of F. For a given value of l, this number can be maximum if the l paths of F occur with smaller indices than those of the $10 - 2l$ edges of the matching. This is due to the fact that the l vertices y_i belonging to paths of length 2 have degree 2 and thus they cannot be adjacent by an edge not in F to any vertex x_i with $i < j$. When this condition is fulfilled, there are at most l^2 edges between the vertices $x_1, x_2, x_3, x_4, ..., x_{2l}$ (since the number of edges of a triangle-free graph of order $2l$ is at most $(2l)^2/4$), $2l(10 - 2l)$ edges of type $x_i y_j$ between the sets $\{x_1, x_2, ..., x_{2l}\}$ and $\{y_{2l+1}, y_{2l+2}, ..., y_{10}\}$, and $\frac{(10-2l)(10-2l-1)}{2}$ edges of type $x_i x_j$ or $x_i y_j$ with $i < j$ between the vertices of the set $\{x_{2l+1}, ..., x_{10}, y_{2l+1}, ..., y_{10}\}$. Then, $\mu \leqslant l^2 + 2l(10 - 2l) + \frac{(10-2l)(10-2l-1)}{2} = 45 - l^2 + l$. Counting the edges of $G - F$ adjacent to some edge of F, we find at least $(10 - 2l)\, \delta(H)$ edges adjacent to an edge of a matching of F and $2l(\delta(H) - 1)$ edges adjacent to an edge of a path of length 2 (since each vertex y_i on such a path has degree 2 in G). At most $45 - l^2 + l$ of these edges have their two endvertices in $P \cup Q$ and are thus counted twice. Hence $m(G) \geqslant (10 - 2l)\, \delta(H) + 2l(\delta(H) - 1) - 45 + l^2 - l + 10$, that is $v(H) = m(G) \geqslant 10\delta(H) + l^2 - 3l - 35 \geqslant 10\delta(H) - 37$ since l is an integer between 0 and 5. This is impossible by the hypothesis $\delta(H) > \frac{v(H) + 37}{10}$, which achieves the proof by contradiction. ∎

Let G_p be the graph obtained from the Petersen graph P by adding p pendant edges at each vertex of P, and let H_p be the line graph of G_p. Then $v(H_p) = m(G_p) = 10p + 15$ and $\delta(H_p) = (p - 1) + 3 = p + 2$ (this is the degree of a vertex of H_p corresponding to a pendant edge of G_p). Hence $10\delta(H_p) = v(H_p) + 5$. The graph G_p is essentially 3-edge-connected and has no dominating circuit since P is not hamiltonian. Therefore H_p is 3-connected and is not hamiltonian. This proves that the lower bound on $\delta(H_p)$ in Theorem 1 to ensure the hamiltonicity of H_p is at least $(v(H_p) + 6)/10$, and that we cannot improve the coefficient $\frac{1}{10}$. However, the additive constant $\frac{37}{10}$ is probably not the best one. In [6], the authors gave an example of a non-hamiltonian 3-connected claw-free graph L of order 20 and minimum degree 3. It can be checked that L is the line graph of the graph obtained from the Petersen graph by subdividing once each edge of a 5-matching.

We conjecture that every 3-connected claw-free graph of order v and minimum degree $\delta \geqslant (v+11)/10$ is hamiltonian.

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